DEFORMATION THEORY, COMPUTATIONS, AND TORIC GEOMETRY

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1. Introduction

Throughout, we will work over an algebraically closed field $\mathbb{K}$ of characteristic zero. All rings will be $\mathbb{K}$-algebras, all schemes will be $\mathbb{K}$-schemes, and all maps will be over $\mathbb{K}$. Good references for this section are [Har10, Ser06]

Let X be a scheme over $\mathbb{K}$. Our motivating question is the following:

Question 1.1. What kind of flat families $\pi : \mathcal{X} \to S$ exist that have X as a fiber?

This gives insight into how $\mathcal{X}$ might fit into a moduli space. Here, *flatness* guarantees that the fibers of π behave nicely. For example, if $\mathcal{X} \subset \mathbb{P}^n \times S$, S is integral, and π is the projection, flatness is equivalent to all geometric fibers having the same Hilbert polynomial.

Example 1.2. Consider the embedded family

$$V(x_1^2 + x_2^2 + x_3^2 + tx_0^2) \subset \mathbb{P}^3 \times \mathbb{A}^1$$

over $S = \mathbb{A}^1$. The fiber over 0 is a singular quadric cone, whereas all other fibers are smooth quadrics.

Answering Question 1.1 is very difficult in general. We will vastly simplify things by only considering $S = \operatorname{Spec} A$, where $A \in \mathbf{Art}$, the category of local Artinian $\mathbb{K}$-algebras with residue field $\mathbb{K}$. Given any local ring R, we will always denote its maximal ideal by $\mathfrak{m}_R$.

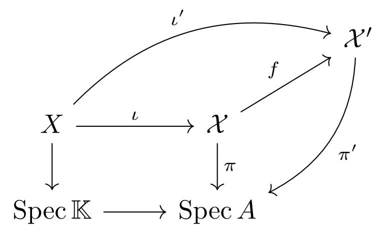
Definition 1.3. A *deformation* of X over $A \in \mathbf{Art}$ is a Cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{\iota} & \mathcal{X} \\ \downarrow & & \downarrow \pi \\ \operatorname{Spec} \mathbb{K} & \longrightarrow & \operatorname{Spec} A \end{array}$$

with π flat. (Cartesian just means that the diagram induces an isomorphism of X with the fiber product $\operatorname{Spec} \mathbb{K} \times_{\operatorname{Spec} A} \mathcal{X}$.) $\mathcal{X}$ is called the *total space* of the deformation and $\operatorname{Spec} A$ the *base*.

We will often abbreviate a diagram as above by just $\pi : \mathcal{X} \to \operatorname{Spec} A$ (or even just $\mathcal{X}$) when the other parts of the diagram are understood. A *morphism* of deformations of X over A from $\pi : \mathcal{X} \to \operatorname{Spec} A$ to $\pi : \mathcal{X}' \to \operatorname{Spec} A$ is a map

$f : \mathcal{X} \to \mathcal{X}'$ such that $\pi = \pi' \circ f$ and $\iota' = f \circ \iota$.



An important observation that we will constantly use is that as topological spaces, X and $\mathcal{X}$ are identical; they only differ in terms of their structure sheaves.

Exercise 1.4. Show that any morphism of deformations $f : \mathcal{X} \to \mathcal{X}'$ is automatically an isomorphism. *Hint: induct on the length of A and use flatness.*

We can now define our main object of study. Let **Set** denote the category of sets.

Definition 1.5. The functor of deformations of X is the covariant functor

$$\mathrm{Def}_X : \mathbf{Art} \to \mathbf{Set}$$

defined on objects by

$$\mathrm{Def}_X(A) = \{\text{Deformations of } X \text{ up to isomorphism}\}$$

and on morphisms by pullback, that is, $f : A \to A'$ maps $\mathcal{X} \to \operatorname{Spec} A$ to

$$\mathcal{X} \times_{\operatorname{Spec} A} \operatorname{Spec} A' \to \operatorname{Spec} A'.$$

Exercise 1.6. Check that Def_X is well-defined for morphisms.

We will occasionally be a bit sloppy and conflate an isomorphism class of deformations and a particular representative of that class; we only do this when it won't lead us into problems.

Example 1.7. $\mathrm{Def}_X(\mathbb{K})$ is the singleton set.

We call any functor $\mathrm{F} : \mathbf{Art} \to \mathbf{Set}$ such that $\mathrm{F}(\mathbb{K})$ is a singleton a *functor of Artin rings*. The *tangent space* to such a functor is $\mathrm{F}(\mathbb{K}[t]/t^2)$.

Example 1.8. Given any local $\mathbb{K}$-algebra R, the functor $\mathrm{Hom}(R, -)$ of local $\mathbb{K}$-algebra homomorphisms from R to a given Artinian ring is a functor of Artin rings.

Exercise 1.9. For R any local $\mathbb{K}$-algebra, the tangent space of $\mathrm{Hom}(R, -)$ is $(\mathfrak{m}_R/\mathfrak{m}_R^2)^*$. *This justifies the terminology.*

Our dream would be for Def_X to be a representable functor. More precisely, let **Comp** be the category of complete local noetherian $\mathbb{K}$-algebras with residue field $\mathbb{K}$. We dream of finding $R \in \mathbf{Comp}$ so that $\mathrm{Hom}(R, -) : \mathbf{Art} \to \mathbf{Set}$ is isomorphic to Def_X. (Even better, we might ask for R to be even more geometric, e.g. the localization at a maximal ideal of a finitely generated $\mathbb{K}$-algebra.) We could then think of $\operatorname{Spec} R$ as being the space parametrizing all possible infinitesimal deformations of X.

Unfortunately, this is often impossible, so we will concentrate on asking for something weaker.

Definition 1.10. A map $\mathrm{F} \to \mathrm{G}$ of functors of Artin rings is *smooth* if for every surjective $A' \to A$ in **Art**, the induced map

$$\mathrm{F}(A') \to \mathrm{G}(A') \times_{\mathrm{G}(A)} \mathrm{F}(A)$$

is surjective.

Why should this notion be called smooth? It is because for representable functors, this is the same thing as a smooth map of rings. We call a surjection of $\mathbb{K}$-algebras $B' \to B$ a *nilpotent extension* if the kernel is nilpotent.

Lemma 1.11 (Infinitesimal lifting lemma, cf. [Ser06, Theorem C9])**.** *Consider a $\mathbb{K}$-algebra homomorphism $f : R \to S$ with S a localization of a finite type R-algebra. The following are equivalent:*

(1) S is a smooth[1] *R-algebra;*
(2) For every nilpotent extension $B' \to B$ of local rings with a commutative square

$$\begin{array}{ccc} S & \longrightarrow & B \\ \uparrow & \searrow & \uparrow \\ R & \longrightarrow & B' \end{array}$$

there exists $S \to B'$ making the resulting diagram commute;
(3) For all primes $\mathfrak{p}$ of S, $\mathrm{Hom}(S_{\mathfrak{p}}, -) \to \mathrm{Hom}(R_{f^{-1}(\mathfrak{p})}, -)$ is a smooth map of functors of Artin rings.

Criterion (3) is just the special case of (2) restricted to extensions of Artinian rings.

Definition 1.12. A *hull* for Def_X is some $R \in$ **Comp** and a smooth map of functors $\mathrm{Hom}(R, -) \to \mathrm{Def}_X$ which is an isomorphism on tangent spaces.

By *Schlessinger's theorem*, Def_X has a hull if X is affine with isolated singularities, or X is proper over $\mathrm{Spec}\,\mathbb{K}$.

What does it mean that $\mathrm{Hom}(R, -) \to \mathrm{Def}_X$ is a hull? For any n, the map $R \to R/\mathfrak{m}_R^n$ gives a deformation $\mathcal{X}_n \in \mathrm{Def}_X(R/\mathfrak{m}_R^n)$. By smoothness (applied to $A = \mathbb{K}$), for any other deformation $\mathcal{Y} \in \mathrm{Def}_X(A')$, for n sufficiently large there is a map $R/\mathfrak{m}_R^n \to A'$ such that

$$\mathcal{Y} \cong \mathcal{X}_n \times_{\mathrm{Spec}\, R/\mathfrak{m}^n} \mathrm{Spec}\, A'.$$

In other words, any deformation of X can be induced from some $\mathcal{X}_n$ by pullback. The information encoded by $\mathrm{Hom}(R, -) \to \mathrm{Def}_X$ is the same as knowing R and the family of deformations $\mathcal{X}_n \in \mathrm{Def}_X(R/\mathfrak{m}_R^n)$ since any $R \to A$ necessarily factors through $R/\mathfrak{m}_R^n$ for n sufficiently large.[2]

Exercise 1.13. Show that if Def_X has a hull, it is unique up to (non-canonical) isomorphism. *Hint: reduce to showing that any surjective endomorphism of an Artinian ring is an isomorphism.*

Exercise 1.14. Show that if Def_X has a hull $\mathrm{Hom}(R, -) \to \mathrm{Def}_X$, it is characterized by the following property: for any $A \in$ **Art** and $\mathcal{Y} \in \mathrm{Def}_X(A')$, there is a map $f : R \to A$ with uniquely determined differential such that $\mathcal{Y} \cong \mathcal{X}_n \times_{\mathrm{Spec}\, R/\mathfrak{m}^n} \mathrm{Spec}\, A'$.

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[1] Often times (as in [Ser06, Appendix C]) 3 above plus essentially of finite type is taken as the definition of smooth. Here, I mean smooth as defined by the usual Jacobian criterion.

[2] R together with the $\mathcal{X}_n$ also go by the name miniversal or semiuniversal deformation.

Our goal is to explicitly describe the hull of Def_X *in concrete situations.* When is this possible? Situations that I know about:

(1) X is given by equations (i.e. X affine or projective). Using the relational criterion of flatness [Ste03, pp 8] one can iteratively lift equations and relations to obtain a hull [Ste95, Ilt12].
(2) X has special structure making Def_X particularly simple, e.g. Def_X is smooth (if X is Fano or Calabi-Yau) or there are only quadratic obstructions (if Def_X is controlled by a "formal" DGLA).
(3) *Our focus*: X is smooth and proper over $\mathbb{K}$.

Example 1.15. For the singular quadric

$$V(x_1^2 + x_2^2 + x_3^2) \subset \mathbb{P}^3,$$

a hull is given by $R = \mathbb{K}[[t]]$ along with the deformations $\mathcal{X}_n$ obtained from Example 1.2 by base changing to $\operatorname{Spec} \mathbb{K}[t]/t^n$.

2. Deformations of Smooth Varieties

A good reference for this section is [Man22]. We now consider the special situation that X is smooth. The motto here is:

Def_X is controlled by the tangent sheaf $\mathcal{T}_X$.

We will make this precise. First we deal with the affine case:

Lemma 2.1. *Suppose X is smooth and affine, and A is in* **Art***. Then any element of* $\operatorname{Def}_X(A)$ *is isomorphic to the product family* $X \times \operatorname{Spec} A$.

Proof. Take $X = \operatorname{Spec} B$, let $\mathcal{X} \in \operatorname{Def}_X(A)$ be given by $\mathcal{X} = \operatorname{Spec} B'$. Applying the second criterion of the infinitesimal lifting lemma to $R = A$ and $S = A \otimes B$, we obtain $B \otimes A \to B'$, that is, a map of deformations $\mathcal{X} \to X \times \operatorname{Spec} A$. This is an isomorphism by Exercise 1.4.

$$\begin{array}{ccc} X \times \operatorname{Spec} A & \longleftarrow & X \\ \downarrow & \nwarrow & \downarrow \\ \operatorname{Spec} A & \longleftarrow & \mathcal{X} \end{array}$$

□

Moving to the non-affine case, suppose we have an affine open cover $\mathcal{U} = \{U_i\}$ of X. Consider any deformation $\mathcal{X} \in \operatorname{Def}_X(A)$ for some $A \in$ **Art**. By Lemma 2.1 we have isomorphisms

$$\phi_i : \mathcal{O}_X(U_i) \otimes A \to \mathcal{O}_{\mathcal{X}}(U_i)$$

and composing the restriction of ϕ_j and ϕ_i^{-1} to $U_{ij} = U_i \cap U_j$ we obtain

$$\phi_{ij} = (\phi_i^{-1})_{|U_{ij}} \circ (\phi_j)_{|U_{ij}} : \mathcal{O}_X(U_{ij}) \otimes A \to \mathcal{O}_X(U_{ij}) \otimes A.$$

These automorphisms ϕ_{ij} are called *infinitesimal automorphisms*.

Definition 2.2. Given a ring R and $A \in$ **Art**, we define $\operatorname{Aut}_R(A)$ to be the set of all A-algebra homomorphisms $\phi : R \otimes A \to R \otimes A$ such that $\phi \otimes A/\mathfrak{m}_A$ is the identity.

Observe that in fact the ϕ_{ij} from above belong to $\operatorname{Aut}_{\mathcal{O}_X(U_{ij})}(A)$. It is straightforward to verify a number of other properties:

(1) After restricting to $U_{ijk} = U_i \cap U_j \cap U_k$ they satisfy the cocycle condition $\phi_{jk}\phi_{ik}^{-1}\phi_{ij} = \mathrm{id}$.
(2) Choosing different ϕ_i' gives us $\sigma_i = \phi_i^{-1}\phi_i' \in \mathrm{Aut}_{\mathcal{O}_X(U_i)}(A)$ satisfying $\phi_i' = \phi_i \circ \sigma_i$. Then $\phi_{ij}' = \sigma_i^{-1}\phi_{ij}\sigma_j$. We thus say that collections of automorphisms $\{\phi_{ij}\}$ and $\{\phi_{ij}'\}$ are *equivalent* if they differ by some $\{\sigma_i\}$ as above.
(3) Isomorphic deformations yield equivalent data $\{\phi_{ij}\}$.
(4) Given a collection of infinitesimal automorphisms $\{\phi_{ij}\}$ satisfying the cocyle condition, one can glue to obtain a corresponding deformation.

Thus we obtain:

Theorem 2.3. *Let X be a smooth variety with open cover $\mathcal{U} = \{U_i\}$. For $A \in \mathbf{Art}$,*

$$\mathrm{Def}_X(A) \cong \{\{\phi_{ij}\} \mid \phi_{ij} \in \mathrm{Aut}_{\mathcal{O}_X(U_{ij})}(A) \text{ satisfy the cocycle condition}\}/\sim$$

where $\sim$ is the equivalence relation from point 2 above.

We would like to *linearize* this description.

Exercise 2.4. Let S be a $\mathbb{K}$-algebra. There is a bijection

$$\mathrm{Der}(S,S) \to \mathrm{Aut}_S(\mathbb{K}[t]/t^2)$$

sending ∂ to $\mathrm{id} + t\partial$.

This generalizes.

Definition 2.5. Let S be a $\mathbb{K}$-algebra and $A \in \mathbf{Art}$. Given $\partial \in \mathrm{Der}(S,S) \otimes \mathfrak{m}_A$, we define

$$e^{\partial} = \sum_{k\geq 0} \frac{1}{k!}\partial^k \in \mathrm{Hom}(S\otimes A, S\otimes A).$$

Since $\mathfrak{m}_A^n = 0$ for $n \gg 0$, the above sum is finite.

Theorem 2.6 ([Man22, Proposition 3.4.3])**.** *The map $e : \mathrm{Der}(S,S) \otimes \mathfrak{m}_A \to \mathrm{Aut}_S(A)$ is an isomorphism. The inverse of e^{∂} is $e^{-\partial}$.*

Proof sketch. To show that $e^{\partial} \in \mathrm{Aut}_S(A)$, use the Leibniz rule (and various identities). To show that the induced map e is an isomorphism, induct on the length of A. □

Notice that in particular, e gives an isomorphism

$$\mathcal{T}_X(U_{ij}) \otimes \mathfrak{m}_A \to \mathrm{Aut}_{\mathcal{O}_X(U_{ij})}(A).$$

Using the exponential map, we can define a binary operation $\star$ on $\mathcal{T}_X(U) \otimes \mathfrak{m}_A$ via the equality

$$e^{x\star y} = e^x e^y.$$

This gives $\mathcal{T}_X(U)\otimes\mathfrak{m}_A$ the structure of a (non-abelian) group. The Baker-Campbell-Hausdorff theorem says that $\star$ can be expressed solely in terms of iterated commutators; the first few terms are

$$x \star y = x + y + \frac{1}{2}[x,y] + \frac{1}{12}[x,[x,y]] - \frac{1}{12}[y,[x,y]] + \dots$$

Using $\star$ and the exponential map, we may reintrepet Theorem 2.3. Recall that the *alternating Čech complex* for a sheaf $\mathcal{F}$ with respect to the cover $\mathcal{U}$ is the complex $\check{C}^\bullet(\mathcal{U},\mathcal{F})$ with

$$\check{C}^k(\mathcal{U},\mathcal{F}) = \{\alpha \in \bigoplus_{i_0,\ldots,i_k} \mathcal{F}(U_{i_0\ldots i_k}) \mid \alpha_{i_0\ldots i_k} = \operatorname{sign}(\sigma)\alpha_{\sigma(i_0\ldots i_k)}\ \forall \sigma \in S_{k+1}\}$$

where the action of the permutation σ is the obvious one, and with differential $d : \check{C}^{k-1}(\mathcal{U},\mathcal{F}) \to \check{C}^k(\mathcal{U},\mathcal{F})$ given by

$$d(\alpha)_{i_0\ldots i_k} = \sum_{j=0}^{k} (-1)^j (\alpha_{i_0\ldots \hat{i_j}\ldots i_k})_{|U_{i_0\cdots i_k}}.$$

Here $\hat{i_j}$ means we remove the index i_j. Of particular note are the differentials d^0 and d^1:

$$d^0(\alpha)_{ij} = \alpha_j - \alpha_i \qquad d^1(\alpha)_{ijk} = \alpha_{jk} - \alpha_{ik} + \alpha_{ij}$$

where we are implicitly restricting sections to U_{ij} and U_{ijk}.

We have that $\operatorname{Def}_X(A)$ may be identified with

$$\left\{\alpha \in \check{C}^1(\mathcal{U},\mathcal{T}_X) \otimes \mathfrak{m}_A \mid \mathfrak{o}(\alpha) = 0\right\}/\sim$$

where

$$\mathfrak{o}(\alpha)_{ijk} = \alpha_{jk} \star (-\alpha_{ik}) \star \alpha_{ij}$$

and $\sim$ is relation induced by $\alpha \sim \alpha'$ if and only if there exists $\gamma \in \check{C}^0(\mathcal{U},\mathcal{T}_X) \otimes \mathfrak{m}_A$ with $\alpha'_{ij} = -\gamma_i \star \alpha_{ij} \star \gamma_j$.

Exercise 2.7. For $A = \mathbb{K}[t]/t^2$, $\star$ is the same as $+$.

Exercise 2.8. Show that $\operatorname{Def}_X(\mathbb{K}[t]/t^2)$ may be identified with

$$\ker d^1 / \operatorname{im} d^0 = \check{H}^1(\mathcal{U},\mathcal{T}_X).$$

Given a surjection $A' \to A$ in **Art** and $\mathcal{X} \in \operatorname{Def}_X(A)$, we would like to know if there is $\mathcal{X}' \in \operatorname{Def}_X(A')$ restricting to $\mathcal{X}$. We will consider this for an extension

$$0 \to I \to A' \to A \to 0$$

with $\mathfrak{m}_{A'} \cdot I = 0$ (sometimes called a *small extension*). Representing $\mathcal{X}$ by

$$\alpha \in \check{C}^1(\mathcal{U},\mathcal{T}_X) \otimes \mathfrak{m}_A$$

satisfying $\mathfrak{o}(\alpha) = 0$, the question becomes: does there exist

$$\alpha' \in \check{C}^1(\mathcal{U},\mathcal{T}_X) \otimes \mathfrak{m}_{A'}$$

satisfying $\alpha' \otimes_{A'} A = \alpha$ such that $\mathfrak{o}(\alpha') = 0$? We call such α' a *lift* of α.

Exercise 2.9. Let α be as above. Take any $\alpha' \in \check{C}^1(\mathcal{U},\mathcal{T}_X) \otimes \mathfrak{m}_{A'}$ such that $\alpha' \otimes_{A'} A = \alpha$. Then $\mathfrak{o}(\alpha')$ is a cocycle in $\check{C}^2(\mathcal{U},\mathcal{T}_X) \otimes I$. Furthermore, α has a lift to A' if and only if the class of $\mathfrak{o}(\alpha')$ in $\check{H}^2(\mathcal{U},\mathcal{T}_X) \otimes I = \ker d^2/\operatorname{im} d^1$ vanishes.

We thus see that the tangent space to Def_X is $\check{H}^1(\mathcal{U},\mathcal{T}_X)$, and $\check{H}^2(\mathcal{U},\mathcal{T}_X)$ may be viewed as an "obstruction space" for Def_X: it detects obstructions to lifting deformations to larger bases.

The construction of this section can be reversed and carried out for any sheaf of Lie algebras $\mathcal{L}$ on X: there is still a BCH product $\star$ and one can define a functor $\mathrm{F}_\mathcal{L}$ of Artin rings via

$$\mathrm{F}_\mathcal{L}(A) = \left\{\alpha \in \check{C}^1(\mathcal{U},\mathcal{L}) \otimes \mathfrak{m}_A \mid \mathfrak{o}(\alpha) = 0\right\}/\sim.$$

The tangent space for this functor is given by $\check{H}^1(\mathcal{U},\mathcal{L})$, and $\check{H}^2(\mathcal{U},\mathcal{L})$ detects obstructions to lifting.

Example 2.10. Let X be a scheme, $\mathcal{E}$ a vector bundle on X. Then the functor of deformations of the vector bundle $\mathcal{E}$ is isomorphic to $\mathrm{F}_{\mathscr{E}nd(E)}$.

3. Solving the Deformation Equation

The basic idea in this section is found in [Ste95]. A more detailed exposition with complete proofs is in [IR24]. As in the previous section, let's assume that X is smooth; we'll additionally assume that X is complete so that $H^i(X,\mathcal{T}_X)$ is finite dimensional for all i. We wish to use the description of Def_X in terms of $\check{C}^\bullet(\mathcal{U},\mathcal{T}_X)$ in order to compute a hull $\mathrm{Hom}(R,-)\to \mathrm{Def}_X$ of Def_X.

To start, fix cocycles $\theta_1,\ldots,\theta_p\in \check{C}^1(\mathcal{U},\mathcal{T}_X)$ whose images give a basis of $H^1(X,\mathcal{T}_X)$ and cocycles $\omega_1,\ldots,\omega_q\in \check{C}^2(\mathcal{U},\mathcal{T}_X)$ whose images give a basis of $H^2(X,\mathcal{T}_X)$. Set $S=\mathbb{K}[[t_1,\ldots,t_p]]$ with maximal ideal $\mathfrak{m}$. We will inductively construct

$$\alpha^{(r)}\in \check{C}^1(\mathcal{U},\mathcal{T}_X)\otimes\mathfrak{m}\qquad g_\ell^{(r)}\in\mathfrak{m}^2,\ \ell=1,\ldots,q$$

starting with

$$\alpha^{(1)}=\sum_{\ell=1}^{p} t_\ell\theta_\ell$$

and $g_1^{(1)}=\ldots=g_q^{(1)}=0$. We can think of $\alpha^{(r)}$ as encoding the rth order approximation of the semiuniversal family $\mathcal{X}_{r+1}$ and the $g_\ell^{(r)}$ as the equations for the rth order approximation of the base space $\mathrm{Spec}\, R/\mathfrak{m}_R^{r+1}$.

Set $J_r=\langle g_\ell^{(r)}+\mathfrak{m}^{r+1}\rangle\subset S$. To construct $\alpha^{(r+1)},g_\ell^{r+1}$ we will solve the *deformation equation*

$$\mathfrak{o}(\alpha^{(r)})-\sum_{\ell=1}^{q} g_\ell^{(r)}\cdot\omega_\ell\equiv d(\beta^{(r+1)})+\sum_{\ell=1}^{q}\gamma_\ell^{(r+1)}\cdot\omega_\ell\qquad \mod \mathfrak{m}\cdot J_r \tag{3.1}$$

for

$$\beta^{(r+1)}\in\check{C}^1(\mathcal{U},\mathcal{T}_X)\otimes\mathfrak{m}^{r+1}\qquad \gamma_\ell^{(r+1)}\in\mathfrak{m}^{r+1},\ \ell=1,\ldots,q.$$

We then set

$$\alpha^{(r+1)}=\alpha^{(r)}-\beta^{(r+1)}\qquad g_\ell^{(r+1)}=g_\ell^{(r)}+\gamma_\ell^{(r+1)}.$$

Proposition 3.2. *It is possible to iteratively solve* (3.1) *for* $\beta^{(r+1)},\gamma_\ell^{(r+1)}$.

Proof sketch. Given a solution of (3.1) modulo $\mathfrak{m}\cdot J_{r-1}$, it follows from properties of $\star$ that

$$\mathfrak{o}(\alpha^{(r)})-\sum_{\ell=1}^{q} g_\ell^{(r)}\cdot\omega_\ell\equiv 0\qquad \mod \mathfrak{m}\cdot J_{r-1}$$

so in particular $\mathfrak{o}(\alpha^{(r)})\equiv 0 \mod J_r$. Considering the small extension

$$0\to J_r\to S/(\mathfrak{m}\cdot J_r)\to S/J_r$$

and using Exercise 2.9 shows that a solution exists. □

Let g_ℓ be the projective limit of $g_\ell^{(r)}$, and α be the projective limit of the $\alpha^{(r)}$. Take

$$J=\langle g_1,\ldots,g_q\rangle\qquad R=S/J\qquad R_n=S/J_n.$$

The cochain α determines a map $\mathrm{Hom}(R,-)\to\mathrm{Def}_X$ as follows: for $A\in\mathbf{Art}$, any $\phi: R\to A$ factors through $R_n\to A$ for $n\gg 0$. The homomorphism ϕ maps to the deformation corresponding under the exponential map to $\phi(\alpha^{(n)})$.

Proposition 3.3. *The above map* $\mathrm{Hom}(R,-)\to\mathrm{Def}_X$ *is a hull.*

Proof sketch. By construction, $\mathrm{Hom}(R,\mathbb{K}[t]/t^2)\to\mathrm{Def}_X(\mathbb{K}[t]/t^2)$ is an isomorphism *(Verify this!)* To show that the map of functors is smooth, by the *standard smoothness criterion* [Man22, Theorem 3.6.5] it suffices to show that there is an "injective map of obstruction spaces". This is guaranteed by the construction. □

In practice, solving (3.1) can be difficult since $\check{C}^1(\mathcal{U},\mathcal{T}_X)$ and $\check{C}^2(\mathcal{U},\mathcal{T}_X)$ are typically very large spaces (i.e. not finite dimensional vector spaces) and not particularly amenable to computation. In the next section we will study a situation where we can overcome this problem by breaking these spaces up into finite dimensional pieces.

Exercise 3.4. Suppose that we have a $\mathbb{K}$-linear map $\psi:\check{C}^2(\mathcal{U},\mathcal{T}_X)\to\check{C}^1(\mathcal{U},\mathcal{T}_X)$ such that for any coboundary $\omega\in\check{C}^2(\mathcal{U},\mathcal{T}_X)$, $d(\psi(\omega))=\omega$.

(1) Setting $\omega_\ell'=\omega_\ell-d(\psi(\omega_\ell))$, show that the images of $\omega_1',\ldots,\omega_q'$ still give a basis for $H^2(X,\mathcal{T}_X)$, and $\psi(\omega_\ell')=0$ for all ℓ.

(2) Assuming now that $\psi(\omega_\ell)=0$ for all ℓ, show that we can solve the deformation equation as follows. Let ξ be the normal form of $\mathfrak{o}(\alpha^{(r)})-\sum_{\ell=1}^q g_\ell^{(r)}\cdot\omega_\ell$ with respect to $\mathfrak{m}\cdot J_r$ for some graded local monomial order.[3]. Then we can take
$$\beta^{(r+1)}=\psi(\xi)$$
and $\gamma_\ell^{(r+1)}$ is determined by
$$\xi-d(\psi(\xi))=\sum\gamma_\ell^{(r+1)}\omega_\ell.$$

4. Deformations of Smooth Toric Varieties

Most of this section is joint work with Sharon Robins [IR24]. Basic references for toric varieties are [Ful93, CLS11].

Definition 4.1. An *toric variety* is a normal variety X equipped with a faithful action of an algebraic torus $T\cong(\mathbb{K}^*)^n$ having a dense orbit in X.

Example 4.2. Projective space $\mathbb{P}^n$ has an obvious torus action and is a toric variety; so do products of projective spaces. The projectivized bundle
$$\mathrm{Proj}_{\mathbb{P}^n}(\mathcal{O}(a_1)\oplus\ldots\oplus\mathcal{O}(a_m))$$
similarly has the structure of a toric variety.

The motto here is:

Toric varieties are completely combinatorial.

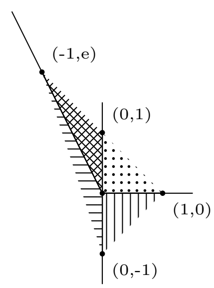
Any toric variety X comes with a canonical open cover $\mathcal{U}$ by T-invariant affine open sets. Furthermore, for any T-invariant affine open set $U\subseteq X$, T acts on
$$H^0(U,\mathcal{T}_X),$$

[3]See e.g. [GP08, Chapter 1]

so the Čech complex decomposes into eigenspaces indexed by characters of the torus. Each graded piece is a complex of finite dimensional $\mathbb{K}$-vector spaces, so solving the deformation equation of the previous section becomes something you can do in practice. Here, we will use the combinatorial structure of X to get an even nicer way to understand Def_X.

The combinatorics takes place in two lattices: the character lattice M of T and its dual $N = \mathrm{Hom}(M, \mathbb{Z})$, the lattice of one-parameter subgroups of T. Throughout, we will identify both lattices with $\mathbb{Z}^n$, with the dual pairing just given by the standard scalar product. The toric variety variety X is completely encoded by a *fan*: a finite set Σ of pointed rational polyhedral cones in $N_\mathbb{R} = N \otimes \mathbb{R}$, closed under taking faces, such that any two cones in Σ intersect in a common face. We write X_Σ for the toric variety corresponding to Σ.

Example 4.3. The fan



corresponds to the eth *Hirzebruch surface* $\mathbb{F}_e = \mathrm{Proj}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(e))$.

One way of understanding the fan Σ associated to a toric variety is that the relative interiors of cones $\sigma \in \Sigma$ contain exactly the one-parameter subgroups of T that have the same limits in X.

Exercise 4.4. Use this to determine the fan for $\mathbb{P}^2$.

There are two more important things to know about the fan Σ:

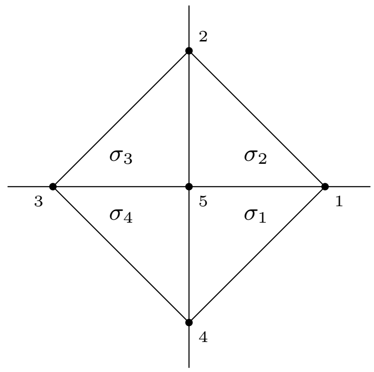
(1) The open sets in the canonical cover $\mathcal{U}$ of X_Σ are in bijection with maximal cones of Σ. Denote the set of maximal cones by $\Sigma_{\max}$, and the open set corresponding to σ by U_σ;
(2) Rays (i.e. one-dimensional cones) of Σ are in bijection with torus invariant divisors of X_Σ. We let $\Sigma(1)$ be the set of rays. For $\rho \in \Sigma(1)$, n_ρ is the primitive element of N generating ρ, and D_ρ is the corresponding divisor. Given $u \in M$, we will write $\rho(u)$ as shorthand for $\langle n_\rho, u\rangle$.

In the remainder of this section, we consider the following running example:

Example 4.5. Fix the lattice $N = \mathbb{Z}^3$. We consider a fan Σ with six rays, where the generator of the ith ray ρ_i is given by the ith column of the following matrix:

$$\begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & -2 & 5 & 1 & -1 \end{pmatrix}.$$

A set of rays belong to a common cone of Σ unless the set contains one of the pairs ρ_1, ρ_3, ρ_2, ρ_4, or ρ_5, ρ_6. An abstract representation of the fan is given by the following figure.



The ray ρ_6 is at ∞, and collections of rays belong to a common cone of Σ exactly when the corresponding vertices in the figure belong to a common simplex. This is the fan for a certain $\mathbb{P}^1$-bundle over $\mathbb{F}_1$.

We know that the cohomology groups of $\mathcal{T}_X$ are important for understanding Def_X; they can be understood combinatorially in this setting:

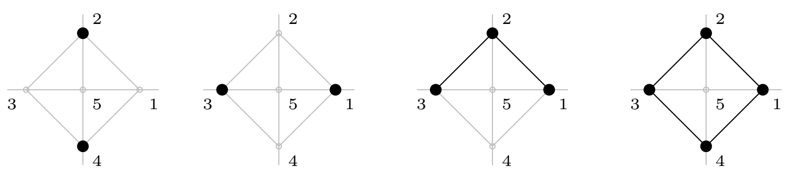
Proposition 4.6. *Let $X = X_\Sigma$ be a smooth complete toric variety. Then for $i \geq 1$,*

$$H^i(X, \mathcal{T}_X) \cong \bigoplus_{u \in M, \rho \in \Sigma(1)} \widetilde{H}^{i-1}(V_{\rho,u}, \mathbb{K})$$

where $V_{\rho,u}$ is the simplicial complex

$$V_{\rho,u} = \bigcup_{\sigma \in \Sigma} \operatorname{conv} \left\{ \rho' \subseteq \sigma \mid \begin{array}{c} \rho'(u) < 0 \textit{ if } \rho' \neq \rho \\ \rho'(u) < -1 \textit{ if } \rho' = \rho \end{array} \right\}.$$

Example 4.7. Continuing the running example, let $u = (0, -2, -1)$ and $v = (-1, 0, 1)$. We obtain the following simplicial complexes:



$V_{\rho_5,u}$ $V_{\rho_6,v}$ $V_{\rho_5,u+v} = V_{\rho_6,u+v}$ $V_{\rho_5,2u+v}$

Verify this! We see that $H^1(X, \mathcal{T}_X)$ is (at least) one-dimensional in degrees u and v, and $H^2(X, \mathcal{T}_X)$ is (at least) one-dimensional in degree $2u+v$. In fact, $H^1(X, \mathcal{T}_X)$ is four-dimensional, and $H^2(X, \mathcal{T}_X)$ is one-dimensional.

We want to describe Def_X in terms of Čech complexes for the simplicial complexes $V_{\rho,u}$. For any $\sigma \in \Sigma_{\max}$, $\sigma \cap V_{\rho,u}$ is either connected or empty, so there a natural surjection

$$\lambda : \mathbb{K} \to H^0(\sigma \cap V_{\rho,u}, \mathbb{K})$$

with unique linear section s. Let $\mathcal{V}_{\rho,u}$ be the closed cover of $V_{\rho,u}$ consisting of $\sigma \cap V_{\rho,u}$ for $\sigma \in \Sigma_{\max}$. This gives us maps

$$\check{C}^\bullet(\Sigma_{\max}, \textstyle\bigoplus_{\rho,u} \mathbb{K}) \xrightarrow{\lambda} \textstyle\bigoplus_{\rho,u} \check{C}^\bullet(\mathcal{V}_{\rho,u}, \mathbb{K}) \quad (\text{with } s \text{ going back})$$

Note that λ is compatible with the Čech differential, but s is not. The vector space $\bigoplus_{\rho,u}\mathbb{K}$ has a Lie bracket given by

$$[\chi^u \cdot f_\rho, \chi^{u'} \cdot f_{\rho'}] := \rho(u')\chi^{u+u'} \cdot f_{\rho'} - \rho'(u)\chi^{u+u'} \cdot f_\rho$$

where e.g. $\chi^u \cdot f_\rho$ identifies that we are in the (ρ, u)th summand. For any $A \in \mathbf{Art}$ this gives a map

$$\mathfrak{o}_\Sigma : \check{C}^0(\Sigma_{\max}, \bigoplus_{\rho,u} \mathfrak{m}_A) \to \check{C}^1(\Sigma_{\max}, \bigoplus_{\rho,u} \mathfrak{m}_A)$$

where

$$\mathfrak{o}_\Sigma(\alpha)_{ij} = -\alpha_i \star \alpha_j.$$

Theorem 4.8. *Let $X = X_\Sigma$ be a smooth complete toric variety. Then* Def_X *is isomorphic to the functor* Def_Σ *defined by*

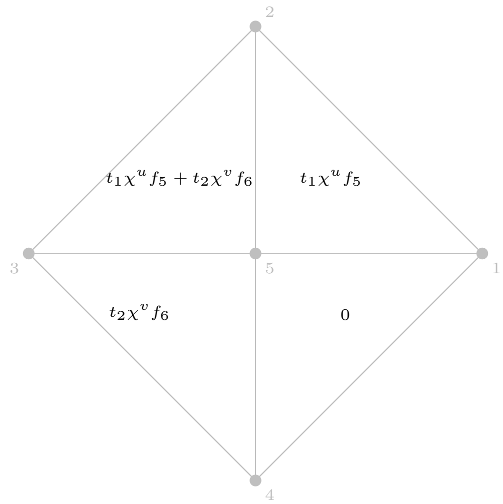
$$\mathrm{Def}_\Sigma(A) = \{\alpha \in \bigoplus_{\rho,u} \check{C}^0(\mathcal{V}_{\rho,u}, \mathfrak{m}_A) \mid \lambda(\mathfrak{o}_\Sigma(s(\alpha))) = 0\}/\sim$$

where $\sim$ is a natural equivalence relation we won't define here.

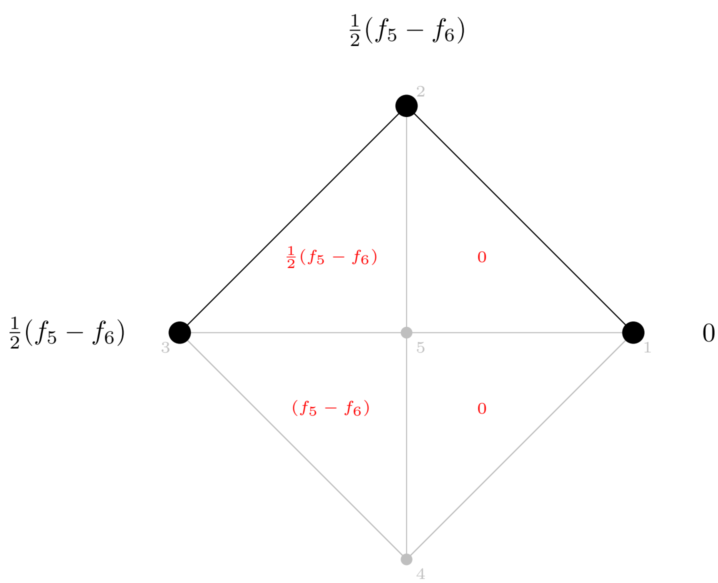
Proof sketch. The toric Euler sequence $\bigoplus \mathcal{O}(D_\rho) \to \mathcal{T}_X$ induces an isomorphism of the deformation functor F controlled by $\bigoplus \mathcal{O}(D_\rho)$ with Def_X. There is a type of "homotopy fiber" construction for the inclusion of sheaves of Lie algebras $\bigoplus_\rho \mathcal{O}(D_\rho) \hookrightarrow \bigoplus_{\rho,u}\mathbb{K}$ that gives an isomorphism of F with Def_Σ. □

This theorem is a big improvement: we get to deal with zero- instead of one-cochains, and we are dealing with locally constant sheaves on simplicial complexes. We can modify the deformation equation (3.1) in an obvious way to construct a hull for Def_Σ. Let's do this for our example! For reasons I won't discuss, we can restrict our attention to the cones $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ (see the figure in Example 4.5). We will ignore the obstruction terms on $\sigma_1 \cap \sigma_3$ and $\sigma_2 \cap \sigma_4$ as these will always vanish. We'll also ignore the two other contributions to $H^1(X, \mathcal{T}_X)$ not mentioned in Example 4.7 as they don't contribute to obstructions. We will always consider obstruction terms on $\sigma_i \cap \sigma_{i+1}$ with indices taken modulo 4.

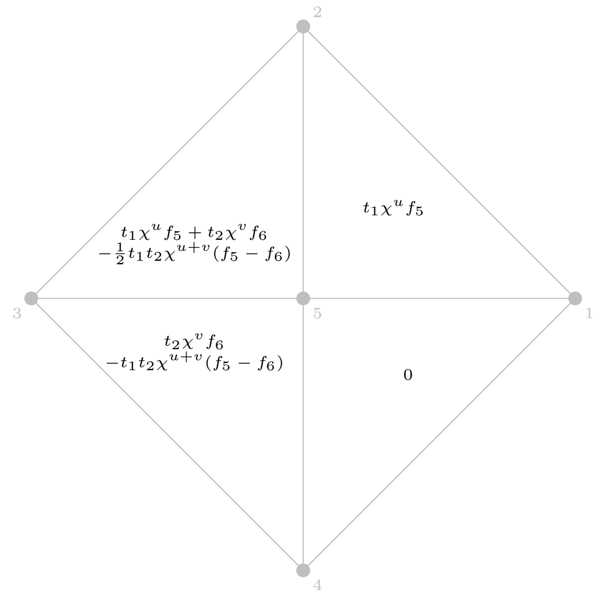
Choosing the images of n_{ρ_2} and n_{ρ_3} as generators of $\widetilde{H}^0(V_{\rho_5,u}, \mathbb{K})$ and $\widetilde{H}^0(V_{\rho_6,v}, \mathbb{K})$ leads to $\alpha^{(1)}$ as pictured:



Using that $\rho_5(u) = \rho_6(v) = -1$ and $\rho_5(v) = \rho_6(u) = 1$, we first compute $\lambda(\mathfrak{o}_\Sigma(s(\alpha^1)))$ modulo $\mathfrak{m}^3$. All terms vanish on the nose, except for the coefficient of t_1t_2 coming from $V_{\rho_5,u+v}$ and $V_{\rho_6,u+v}$, shown in black:



This is the image of the zero-cochain shown above in red. This leads to $g_1^{(2)} = 0$, and $\alpha^{(2)}$ as pictured:

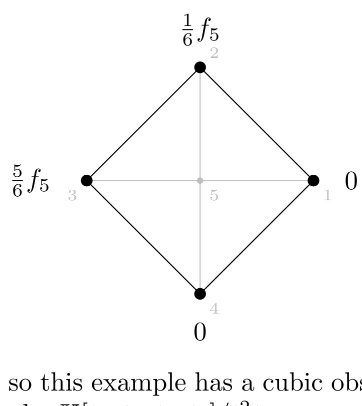


We now compute the coefficient of $t_1^2 t_2$ in $\lambda(\mathfrak{o}_\Sigma(s(\alpha^{(2)})))$. We start with the $2,3$ term. Dropping s from notation for simplicity, we have:

$$\begin{aligned}
[-\alpha_2^{(2)}, \alpha_3^{(2)}] &= t_1t_2\chi^{u+v}(f_5 - f_6) + t_1^2t_2\chi^{2u+v}f_5 + \dots \\
\frac{1}{12}[-\alpha_2^{(2)}, [-\alpha_2^{(2)}, \alpha_3^{(2)}]] &= -\frac{1}{6}t_1^2t_2\chi^{2u+v}f_5 + \dots \\
-\frac{1}{12}[\alpha_3^{(2)}, [-\alpha_2^{(2)}, \alpha_3^{(2)}]] &= -\frac{1}{6}t_1^2t_2\chi^{2u+v}f_5 + \dots
\end{aligned}$$

$$\begin{aligned}
\lambda(\mathfrak{o}_\Sigma(s(\alpha^2))) &= \frac{1}{2}\cdot t_1^2t_2\chi^{2u+v}f_5 - \frac{1}{6}t_1^2t_2\chi^{2u+v}f_5 - \frac{1}{6}t_1^2t_2\chi^{2u+v}f_5 + \dots \\
&= \frac{1}{6}t_1^2t_2\chi^{2u+v}f_5 + \dots.
\end{aligned}$$

Similarly, one computes that the $3,4$ term has a coefficient of $(5/6)\chi^{2u+v}f_5$. We can picture $\lambda(\mathfrak{o}_\Sigma(s(\alpha^{(2)})))$ in $V_{\rho_5, 2u+v}$:



This is not a coboundary, so this example has a cubic obstruction. In fact, one can show that the hull is given by $\mathbb{K}[t_1, t_2, t_3, t_4]/t_1^2t_2$.

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