Automorphisms on deformation families of irreducible holomorphic symplectic manifolds

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joint work with C. Camere

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Aim

- The aim of the talk is to provide a lattice-theoretical classification of non-symplectic automorphisms of prime order acting on irreducible holomorphic symplectic manifolds deformation equivalent to $\mathrm{Hilb}^n(K3)$.
- We will extend results by Boissière, Camere and Sarti (2014), who classified non-symplectic automorphisms in the case $n = 2$.

Introduction

Definition

A complex manifold X is IHS if it is smooth, compact, Kähler with $\pi_1(X) = \{1\}$ and $H^{2,0}(X) = \mathbb{C}\omega_X$, where ω_X is an everywhere non-degenerate holomorphic 2-form.

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Examples

- $\dim(X) = 2$: $K3$ surfaces;
- $\dim(X) = 2n > 2$:
  - $K3^{[n]} = \mathrm{Hilb}^n(K3)$, Hilbert scheme of n points on a $K3$ surface;
  - generalized Kummer variety $K_{n+1}(A)$, for A complex 2-torus;
- $\mathrm{OG6}, \mathrm{OG10}$.

Definition

Let X be IHS; a (biholomorphic) automorphism $\sigma \in \mathrm{Aut}(X)$ of finite order m is purely non-symplectic if $\sigma^*(\omega_X) = \xi_m \omega_X$, with ξ_m primitive m^{th}-root of unity.

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We will consider $X \overset{\text{def}}{\sim} K3^{[n]}$ (we say: X of $K3^{[n]}$-*type*) and $\sigma \in \mathrm{Aut}(X)$ non-symplectic automorphism of odd prime order $3 \leq p \leq 23$.

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Case $p = 2$ already covered by BCS for $n = 2$; for $n \geq 3$ work by Joumaah and Camere–Cattaneo–C.

Lattice classification

For X of $K3^{[n]}$-type, $H^2(X,\mathbb{Z})$ admits a lattice structure:

$$\left(H^2(X,\mathbb{Z}), q_{BBF}\right) \cong L_n := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2(n-1) \rangle.$$

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The non-symplectic automorphism σ of prime order p acts on $H^2(X,\mathbb{Z})$, with:

- invariant lattice $T := \left\{ v \in H^2(X,\mathbb{Z}) : \sigma^*(v) = v \right\} \subset \mathrm{NS}(X)$
- co-invariant lattice $S := T^\perp \cap H^2(X,\mathbb{Z}) \supset \mathrm{Trans}(X)$.

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Plan

- Provide the tools to create, for each value of $n \geq 2$, a list of all possible pairs of lattices (T, S);
- present constructions of automorphisms which realize these admissible pairs.

Properties of T, S (Beauville; Boissière–Nieper-Wisskirchen–Sarti; Tari)

- $\mathrm{rank}(S) = (p-1)m$, with $m \geq 1$
- $\mathrm{sign}(S) = (2, (p-1)m - 2)$
- $\mathrm{sign}(T) = (1, 22 - (p-1)m)$
- $\frac{H^2(X,\mathbb{Z})}{T \oplus S} \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a}$, with $a \geq 0$
- $a \leq m$

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If $2(n-1) = p^\alpha \beta$, $\alpha \geq 0$ and $(p, \beta) = 1$, then $A_{L_n} \cong \frac{\mathbb{Z}}{p^\alpha \mathbb{Z}} \oplus \frac{\mathbb{Z}}{\beta \mathbb{Z}}$.

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Theorem (Camere–C.)

Let $X \sim K3^{[n]}$, $\sigma \in \mathrm{Aut}(X)$ non-symplectic automorphism of prime order $p \neq 2$. Then one of the following holds:

- $A_S \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}} \right)^{\oplus a}$, $A_T \cong A_S \oplus A_{L_n} \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}} \right)^{\oplus a} \oplus \frac{\mathbb{Z}}{2(n-1)\mathbb{Z}}$;
- $\alpha = 1$, $a = 0$, $A_S \cong \frac{\mathbb{Z}}{p\mathbb{Z}}$, $A_T \cong \frac{\mathbb{Z}}{\beta\mathbb{Z}}$;
- $\alpha \geq 1$, $a \geq 1$, $A_S \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}} \right)^{\oplus a+1}$, $A_T \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}} \right)^{\oplus a-1} \oplus \frac{\mathbb{Z}}{2(n-1)\mathbb{Z}}$.

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- $\alpha \geq 1$, $a \geq 1$, $A_S \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a+1}$, $A_T \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a-1} \oplus \frac{\mathbb{Z}}{2(n-1)\mathbb{Z}}$.

Remark: S is always p-elementary, i.e. A_S is of p-torsion.

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Definition

For a given n, a triple of integers (p, m, a) with:

- $3 \leq p \leq 23$ *prime;*
- $m \geq 1$, $(p-1)m \leq 22$*;*
- $0 \leq a \leq \min\{m, 23-(p-1)m\}$

is admissible if there exist $T, S \subset L_n$ orthogonal sublattices such that:

- $\frac{L_n}{T \oplus S} \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a}$*;*
- $\mathrm{sign}(T) = (1, 22-(p-1)m)$, $\mathrm{sign}(S) = (2, (p-1)m-2)$*;*
- A_T, A_S *as in the previous theorem.*

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Theorem (Camere–C.)

If $p^2 \nmid 2(n-1)$, an admissible triple (p, m, a) determines (up to isometries) the p-elementary lattice S. Moreover, if $O(S) \to O(q_S)$ is surjective and $l(A_T) \leq 21 - (p-1)m$, the lattice T is also uniquely determined by the triple.

Using these classification results, we can compile lists of all admissible triples (p, m, a) and the corresponding lattices (T, S), which are unique for any odd p when $n < 10$.

$n = 2$: Boissière–Camere–Sarti (2014)

$n = 3, 4$: Camere–C.

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$n = 3, 4$: Camere–C.

p	m	a	S	T
3	11	0	$U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus A_2(-1)$	$\langle 2 \rangle$
3	10	0	$U^{\oplus 2} \oplus E_8(-1)^{\oplus 2}$	$U \oplus \langle -6 \rangle$
3	10	1	$U \oplus U(3) \oplus E_8(-1)^{\oplus 2}$	$U \oplus \langle -6 \rangle$
3	10	2	$U \oplus U(3) \oplus E_8(-1)^{\oplus 2}$	$U(3) \oplus \langle -6 \rangle$
3	10	3	$U(3)^{\oplus 2} \oplus E_8(-1)^{\oplus 2}$	$U(3) \oplus \langle -6 \rangle$
3	9	1	$U^{\oplus 2} \oplus E_6(-1) \oplus E_8(-1)$	$U \oplus A_2(-1) \oplus \langle -6 \rangle$

⋮

Table: $n = 4$, *order* $p = 3$ *(partial: 46 cases in total)*

Existence of automorphisms

We have found a list of candidates for the invariant and co-invariant lattices $T, S \subset H^2(X, \mathbb{Z})$ of a non-symplectic automorphism of prime order $p \geq 3$ on a manifold $X \sim K3^{[n]}$, with $\mathrm{rank}(S) = (p-1)m$ and $\frac{H^2(X,\mathbb{Z})}{T \oplus S} \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus a}$.

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Tools

- Natural automorphisms on $\mathrm{Hilb}^n(K3)$
- Induced automorphisms on moduli spaces of sheaves on $K3's$
- Torelli-type arguments
- Geometric constructions

Natural automorphisms

Let Σ be a $K3$ surface; $f \in \mathrm{Aut}(\Sigma)$ non-symplectic induces $f^{[n]} \in \mathrm{Aut}(\Sigma^{[n]})$, $f^{[n]}([\eta]) := [f(\eta)] \in \Sigma^{[n]}$, still non-symplectic.

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If $T_f, S_f \subset H^2(\Sigma, \mathbb{Z})$ are the inv. and co-inv. for the action of f:

$$T_{f^{[n]}} \cong T_f \oplus \langle -2(n-1) \rangle, \qquad S_{f^{[n]}} \cong S_f.$$

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Thus, all admissible triples (p, m, a) defining (T, S) of the form $T = T_{K3} \oplus \langle -2(n-1) \rangle$, $S = S_{K3}$, for (T_{K3}, S_{K3}) invariant and co-invariant of a non-symplectic automorphism of order p on a $K3$ surface, are realized by natural automorphisms.

Induced automorphisms

Σ projective $K3$; $H^*(\Sigma, \mathbb{Z}) = H^0(\Sigma, \mathbb{Z}) \oplus H^2(\Sigma, \mathbb{Z}) \oplus H^4(\Sigma, \mathbb{Z})$.
For $\mathcal{F} \in D^b(\Sigma)$, we define the Mukai vector of $\mathcal{F}$:

$$v(\mathcal{F}) := (\mathrm{rk}(\mathcal{F}), c_1(\mathcal{F}), \mathrm{rk}(\mathcal{F}) + \mathrm{ch}_2(\mathcal{F})) \in H^*(\Sigma, \mathbb{Z}).$$

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Take

- $v \in H^*(\Sigma, \mathbb{Z}) \cong \Lambda := U^{\oplus 4} \oplus E_8(-1)^{\oplus 2}$ a positive, primitive vector;
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The coarse moduli space $M_\tau(v)$ of τ-stable objects on Σ of Mukai vector v is a manifold of $K3^{[n]}$-type, with $n = \frac{v^2+2}{2}$ (Yoshioka).

Canonical embedding: $j : H^2(M_\tau(v), \mathbb{Z}) \cong L_n \hookrightarrow \Lambda$, such that $j(H^2(M_\tau(v), \mathbb{Z})) = v^\perp$.

$f \in \mathrm{Aut}(\Sigma)$ induces $\hat{f} \in \mathrm{Aut}(M_\tau(v))$ for a suitable choice of τ, v.

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Theorem (Mongardi–Wandel; 2014)

If $T_f, S_f \subset H^2(\Sigma, \mathbb{Z})$ are the invariant and co-invariant lattices for the action of f, then:

$$T_{\hat{f}} = v^\perp \cap \left(H^0(\Sigma, \mathbb{Z}) \oplus T_f \oplus H^4(\Sigma, \mathbb{Z})\right) \cong v^\perp \cap (T_f \oplus U)$$
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As in the natural case, since the pairs (T_f, S_f) are completely classified, we can easily determine which admissible pairs (T, S) can be realized by induced automorphisms.

Remarks:

- Induced automorphisms generalize natural ones, since $\Sigma^{[n]} \cong M_\tau(v)$ for $v = (1, 0, 1-n) \in H^*_{\mathrm{alg}}(\Sigma, \mathbb{Z})$ and τ v-generic.

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- It is possible to further generalize this construction, considering induced automorphisms on moduli spaces of twisted sheaves $M_v(\Sigma, \alpha)$, for $\alpha \in \mathrm{Br}(\Sigma)$ (Camere–Kapustka–Kapustka–Mongardi).

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- It is possible to further generalize this construction, considering induced automorphisms on moduli spaces of twisted sheaves $M_v(\Sigma, \alpha)$, for $\alpha \in \mathrm{Br}(\Sigma)$ (Camere–Kapustka–Kapustka–Mongardi).
- Using natural and (possibily twisted) induced automorphisms, we can realize all admissible pairs (T, S) of the classification for $n = 2, 3, 4$, except for those where $\mathrm{rank}(T) = 1$.

Special case: $\mathrm{rank}\, T = 1$

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Theorem (Camere–C.)

$2(n-1) = p^\alpha \beta$, with $(p, \beta) = 1$. A triple (p, m, a) with $(p-1)m = 22$ is admissible if and only if $\alpha \in \{0, 1\}$, $a = 1 - \alpha$, $-p$ quadratic residue mod $\frac{4(n-1)}{p^\alpha}$. Only four possible cases:

- $\alpha = 0, (3, 11, 1) : T = \langle 6(n-1) \rangle, S = U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus A_2(-1)$;
- $\alpha = 1, (3, 11, 0) : T = \langle \frac{2(n-1)}{3} \rangle, S = U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus A_2(-1)$;
- $\alpha = 0, (23, 1, 1) : T = \langle 46(n-1) \rangle, S = U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus K_{23}$;
- $\alpha = 1, (23, 1, 0) : T = \langle \frac{2(n-1)}{23} \rangle, S = U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus K_{23}$.

If one of these triples (p, m, a) is admissible for n, it is realized by some non-symplectic automorphism of order p on a manifold of $K3^{[n]}$-type.

Geometric constructions

Let $\mathcal{C}$ be the 10-dim family of cubic 4-folds $Y \subset \mathbb{P}^5$ of equation

$$Y : x_5^3 + F_3(x_0 : \ldots : x_4) = 0, \quad \deg(F_3) = 3.$$

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$n = 2$, $(p, m, a) = (3, 11, 1)$ (Boissière–Camere–Sarti, 2014):
Let $F(Y)$ be the Fano variety of lines on $Y \in \mathcal{C}$: manifold of $K3^{[2]}$-type (Beauville–Donagi, 1985).

σ induces a non-symplectic automorphism $\tilde{\sigma} \in \mathrm{Aut}(F(Y))$ of order 3, whose invariant lattice is $T \cong \langle 6 \rangle$.

$n = 4$, $(p, m, a) = (3, 11, 0)$:

$Y \in \mathcal{C}$ not containing a plane; $\mathrm{Hilb}^{gtc}(Y) \subset \mathrm{Hilb}^{3t+1}(Y)$ Hilbert scheme of generalized twisted cubics on Y.

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Lehn–Lehn–Sorger–van Straten (2013):

$$\begin{array}{ccccc} \mathrm{Hilb}^{gtc}(Y) & \xrightarrow{a} & Z'_Y & \xrightarrow{\tau} & Z_Y \\ \{\text{non-CM curves}\} & \longrightarrow & D & \longrightarrow & j(Y) \end{array}$$

with $a : \mathrm{Hilb}^{gtc}(Y) \to Z'_Y$ a $\mathbb{P}^2$-bundle and $\tau : Z'_Y \to Z_Y$ blow-up along the image of a closed Lagrangian embedding $j : Y \hookrightarrow Z_Y$. Moreover, Z_Y is IHS and $\dim Z_Y = 8$.

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Lehn–Lehn–Sorger–van Straten (2013):

$$\begin{array}{ccccc} \mathrm{Hilb}^{gtc}(Y) & \xrightarrow{a} & Z'_Y & \xrightarrow{\tau} & Z_Y \\ \{\text{non-CM curves}\} & \longrightarrow & D & \longrightarrow & j(Y) \end{array}$$

with $a : \mathrm{Hilb}^{gtc}(Y) \to Z'_Y$ a $\mathbb{P}^2$-bundle and $\tau : Z'_Y \to Z_Y$ blow-up along the image of a closed Lagrangian embedding $j : Y \hookrightarrow Z_Y$. Moreover, Z_Y is IHS and $\dim Z_Y = 8$.

Addington–Lehn (2015): $Z_Y \sim K3^{[4]}$.

Proposition

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Theorem (Camere–C.)

The deformation family $\{(Z_Y, \bar{\sigma}) \mid Y \in \mathcal{C}$ not containing a plane$\}$ is still 10-dimensional and the invariant lattice of $\bar{\sigma}$ is $T \cong \langle 2 \rangle$.