Reduced vs Cuspidal invariants for the quintic 3-fold

joint with L.Battistella, C.Manolache

Workshop in Deformation Theory III

February 20th, 2018

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- Konsetvich's moduli space of stable maps and classical Gromov-Witten;
- Reduced genus 1 Gromov-Witten invariants and Li-Zinger formula;
- Cuspidal Gromov-Witten invariants and relations to the reduced ones for the quintic 3-fold;

Part 1

Recall that:

$$\overline{\mathcal{M}}_{g,n}(X,\beta) = \left\{ f : C \to X \;\middle|\; \begin{array}{c} C \text{ at worst nodal of genus } g \\ (p_1,\ldots,p_n)\in C \text{ smooth (markings)} \\ f_*[C]=\beta \\ + \text{ stability condition} \end{array} \right\} \Big/ \sim$$

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Where the stability condition is:

- if f is constant on a rational component $R \subset C$, then R has at least 3 special points (marking or nodes) ;
- if f is constant on a genus 1 component $E \subset C$, then E has at least 1 special point (marking or nodes) ;

Theorem[Konsetvich, Behrend-Fantechi]

$\overline{\mathcal{M}}_{g,n}(X,\beta)$ is a proper Deligne-Mumford stack and it admits a virtual fundamental class

$$[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir} \in A_{virdim}(\overline{\mathcal{M}}_{g,n}(X,\beta))$$

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For some choice of $\alpha_i \in H^*(X,\mathbb{Z})$, we define the Gromov-Witten invariants:

$$GW_{g,\beta}(X)(\alpha_1,\ldots,\alpha_n) = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}} ev_1^*\alpha_1 \cup \cdots \cup ev_n^*\alpha_n,$$

where

$$ev_i\colon \overline{\mathcal{M}}_{g,n}(X,\beta) \to X, \quad \text{is defined by } \; ev_i(f) = f(p_i).$$

The expected dimension of $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is the one suggested by deformation theory:

$$\begin{aligned} virdim &= \overbrace{3g-3+n}^{\substack{\text{def}-\text{aut} \\ of\ (C,p_i)}} + \overbrace{H^0(C,f^*T_X) - H^1(C,f^*T_X)}^{\substack{\text{def}-\text{obs} \\ \text{fixed } C}} \\ &= 3g-3+n+dim(X)(1-g)+c_1(-K_X)\cdot\beta \\ &= (1-g)(3-dim(X))+c_1(-K_X)\cdot\beta+n \end{aligned}$$

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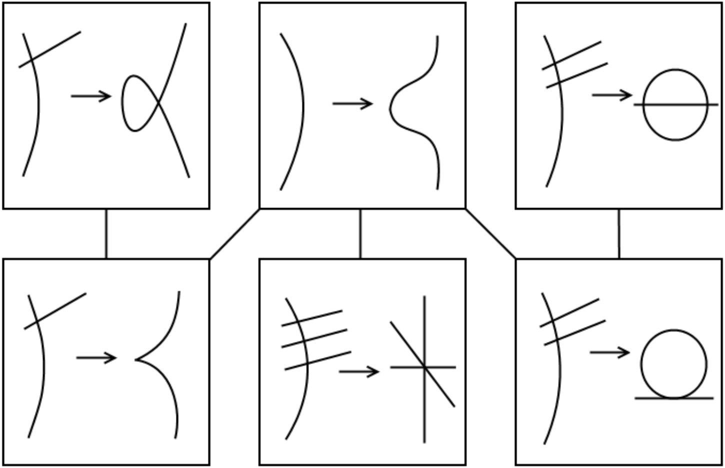
Example

We want to understand the irreducible components of $\overline{\mathcal{M}}_{1,0}(\mathbb{P}^2,3)$

From now on $g = 1$. The following picture has been taken from Viscardi's paper:



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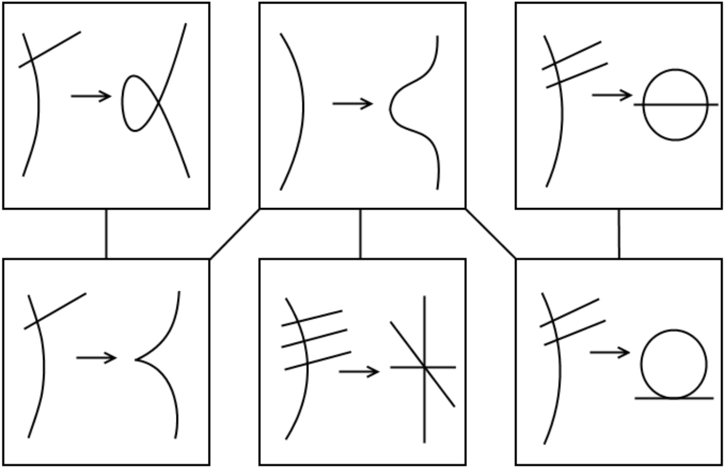

Difficulties of the theory

- The virtual class $[\overline{\mathcal{M}}_{1,n}(X,\beta)]^{vir}$ can be supported on the boundary and this affects the enumerative meaning of the invariants;
- We don't understand well, in general, how $[\overline{\mathcal{M}}_{1,n}(X,\beta)]^{vir}$ splits on the various components so we can't easily discard the boundary contribution;
- For $\mathbb{P}^n$ we can at least compute the invariants using Graber-Pandariphande localisation technique but $X \subseteq \mathbb{P}^n$ is not accessible.

Part 2: Li-Zinger reduced invariants

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Let $X \subseteq \mathbb{P}^n$ be the hypersurface cut out by a section s of $\mathcal{O}_{\mathbb{P}^n}(a)$, then $\overline{\mathcal{M}}_{1,n}(X,\beta) \subseteq \overline{\mathcal{M}}_{1,n}(\mathbb{P}^n,\beta)$ is the locus:

$$\left\{[C,f] \in \overline{\mathcal{M}}_{1,n}(\mathbb{P}^n,\beta) \mid 0 = s \circ f \in H^0(C, f^*\mathcal{O}_{\mathbb{P}^n}(a))\right\}$$

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On the smooth locus this can be described as the section of a vector bundle over the space of maps in $\mathbb{P}^n$:

$$\begin{array}{ccccc} \pi_* f^*\mathcal{O}_{\mathbb{P}^n}(a) & & \mathcal{C} & \xrightarrow{\;f\;} & \mathbb{P}^r \\ \searrow \;\; \nwarrow S & & \downarrow \pi & & \\ & & \overline{\mathcal{M}}_{1,n}(\mathbb{P}^n,\beta) & & \end{array}$$

where $S([C,f]) = [s\circ f]$.

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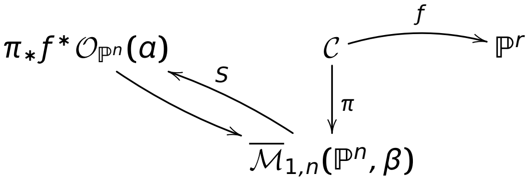

where $S([C,f]) = [s \circ f]$.
We would like to have

$$[\overline{\mathcal{M}}_{1,n}(X,\beta)]^{vir} = c_{top}(\pi_* f^* \mathcal{O}_{\mathbb{P}^n}(a)) \cap [\overline{\mathcal{M}}_{1,n}(\mathbb{P}^n,\beta)]^{vir}$$

Li-Zinger-Vakil successive blow up the moduli space $\mathfrak{M}_{1,n}^{wt,st}$ of pre-stable curves underlying the stable maps moduli space:

$$\overline{\mathcal{M}}_{1,n}(\mathbb{P}^n, \beta) \to \mathfrak{M}_{1,n}^{wt,st}$$

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Theorem[Li-Zinger;Chang-Li]

Let X be the quintic 3-fold in $\mathbb{P}^4$ then

$$GW_1(X) = GW_1^{red}(X) + \frac{1}{12} GW_0(X)$$

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Where the stability condition is:

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- if f is constant on a genus 1 component $E \subset C$, **then** E **has at least** 2 **special point (marking or nodes)**

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$$GW^{cusp}_{1,\beta}(X)(\alpha_1,\ldots,\alpha_n) = \int_{[\overline{\mathcal{M}}_{1,n}(X,\beta)(1)]^{vir}} ev_1^*\alpha_1 \cup \cdots \cup ev_n^*\alpha_n,$$

Looking at the Li-Zinger formula we guessed:

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idea of the proof

- **Proposition[B.,M.,-]** There exists a well defined map of algebraic stacks which contracts elliptic tails and is an isomorphism elsewhere:

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- The fiber product $\mathcal{Z} = \overline{\mathcal{M}}_{1,n}(X,\beta)(1) \times_{\mathfrak{M}_{1,n}^{wt=d}(1)} \mathfrak{M}_{1,n}^{wt=d}$ is a closed sub-stack of $\overline{\mathcal{M}}_{1,n}(X,\beta)$
- Apply Chang-Li technique

- Define the moduli space $\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4, d)^p$ parametrizing $[C, f, \rho]$ where $[C, f]$ is a stable map to $\mathbb{P}^4$ and $\rho \in H^0(C, f^*\mathcal{O}_{\mathbb{P}^4}(-5) \otimes \omega_C)$;

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- p-fields give the same invariants as the quintic up to a sign, :

$$deg[\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4, d)^p]^{vir}_{loc} = (-1)^{5d} deg[\overline{\mathcal{M}}_1^{(1)}(X, d)]^{vir}.$$

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- $\mathcal{Z}^p = \overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4, d)^p \times_{\mathfrak{M}_{1,n}^{wt=d}(1)} \mathfrak{M}_{1,n}^{wt=d}$ and $\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4, d)^p$ are virtually birational:

$$deg[\mathcal{Z}^p]^{vir}_{loc} = deg[\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4, d)^p]^{vir}_{loc}.$$

- The desingularisation $\widetilde{\mathcal{Z}}^p \to \mathcal{Z}^p$ does not alter the invariants,

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- The main component of $\widetilde{\mathcal{Z}}^p$ contributes with the reduced invariants up to a sign, while the boundary is numerically irrelevant:

$$deg[\widetilde{\mathcal{Z}}^p]^{vir}_{loc} = \\ (-1)^{5d} deg\Big(c_{\mathrm{top}}(\tilde{\pi}_* \tilde{f}^* \mathcal{O}_{\mathbb{P}^4}(5) \cap [\widetilde{\mathcal{M}}_1(\mathbb{P}^4, d)^{main}]\Big).$$

Thanks for your attention!